ELASTOPLASTIC TORSION OF ECCENTRIC PIPES

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The torsion of ideal elastic–plastic rods was considered in [1-5]. The solution of elastic–plastic problems by the method of perturbations was examined in researches whose discussion can be found in [6]. Methods developed in [6] are applicable to the solution of problems for doubly connected domains only when one of the contours is completely enclosed by the plastic zone. A modification of the perturbation method is proposed below, which will permit consideration of the development of plastic zones with only a part of the contour enclosed by a plastic zone, hence the idea of expanding the solution in the loading parameter is used [7].

1. The stresses in the torsion of elastic–plastic pipes in an elastic domain are expressed in terms of the stress function by means of the formulas

$$\tau_r = (1/r)\partial u/\partial\theta,\ \tau_\theta = -\partial u/\partial r. \tag{1.1}$$

The stress function u satisfies the equation

$$\Delta u = -2G\omega, \tag{1.2}$$

where G is the shear modulus, and ω is the angle of twist. The flow condition

$$(\partial u/\partial r)^2 + (1/r^2)(\partial u/\partial\theta)^2 = K^2 \tag{1.3}$$

is satisfied in the plastic domain, and we have on the boundary of the contours

$$du/ds = 0. \tag{1.4}$$

Here u and du/dn should be continuous on the elastic–plastic boundary. For a simply connected domain, conditions (1.3) and (1.4) uniquely define the stress state by means of a given torsion. For a multiconnected domain we obtain by integrating (1.4)

$$u = c_p, \tag{1.5}$$

where c_p is constant on each contour, but can only be set equal to zero on one of its contours. To determine the value of c_p from (1.5) the analog of the Bredt theorem in elastic–plastic bodies, which is formulated in [8], must be used.

2. Let us consider the torsion of an eccentric pipe. The contour of the pipe cross section (Fig. 1) are given by the equations

$$L_1 : r = \delta\cos\theta + \sqrt{r_1^2 - \delta^2\sin^2\theta} \approx r_1 + \delta\cos\theta - \frac{\delta^2}{2r_1}\sin^2\theta,\ L_2 : r = r_2. \tag{2.1}$$

The solution of the elastic problem is obtained in [9, 10]. The stress function can be obtained in the elastic domain by using the perturbation method. Limiting ourselves to powers no higher than δ^2, we have

$$u_0 = G\omega\left(\frac{r_2^2 - r^2}{2} + \delta K_1\left(\frac{r_2^2}{r} - r\right)\cos\theta + \delta^2 K_2\left(\frac{r_2^4}{r^2} - r^2\right)\cos 2\theta\right), \tag{2.2}$$

where

$$K_1 = \frac{r_1^2}{r_2^2 - r_1^2};\ K_2 = \frac{r_1^2 r_2^2}{(r_2^2 - r_1^2)(r_2^4 - r_1^4)}.$$

The maximal stress holds at the point A; we obtain from (2.2)

$$\tau_{\max} = G\omega(r_2 + 2\delta K_1 + 4\delta^2 K_2 r_2). \tag{2.3}$$

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The stresses at the point A reached the flow limit for

$$\omega = \omega_0 = KG^{-1}(r_2 + 2\delta K_1 + 4\delta^2 K_2 r_2)^{-1}. \tag{2.4}$$

For $\omega > \omega_0$ a plastic domain forms around the point A in which

$$\tau_r = 0,\ \tau_\theta = K,\ u = K(r_2 - r). \tag{2.5}$$

is satisfied jointly with (1.3). The solution (2.5) will hold for $r_2 - \rho(\theta, \omega) \le r \le r_2$, where $\rho(\theta, \omega)$ is the thickness of the plastic zone along the normal to the contour L_2. Let us introduce the small parameter

$$\varepsilon^2 = (\omega - \omega_0)\,\omega_0^{-1} \tag{2.6}$$

and we seek the solution in the elastic domain in the form of an expansion in ε:

$$u = \sum_{n=0}^{\infty} u_n \varepsilon^n. \tag{2.7}$$

On the elastic–plastic boundary

$$u(r_2 - \rho,\ \theta) = K\rho,\ \partial u/\partial r = -K. \tag{2.8}$$

Taking account of (2.6), Eq. (1.2) will be satisfied if

$$\Delta u_0 = -2G\omega_0,\ \Delta u_1 = 0,\ \Delta u_2 = -2G\omega_0,\ \Delta u_k = 0\ (k = 3, 4 \ldots). \tag{2.9}$$

Using (2.7), the boundary conditions (2.8) can be written in the form

$$\sum_{n=0}^{\infty} u_n (r_2 - \rho,\ \theta)\ \varepsilon^n = K\rho,\ \sum_{n=0}^{\infty} \frac{\partial u_n (r_2 - \rho,\ \theta)}{\partial r} \varepsilon^n = -K. \tag{2.10}$$

Expanding the left side of (2.10) in a series in powers of

$$\begin{gathered}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\partial^m u_n (r_2,\ \theta)}{\partial r^m} \varepsilon^n \rho^m = K\rho,\\ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\partial^{m+1} u_n (r_2,\ \theta)}{\partial r^{m+1}} \varepsilon^n \rho^m = -K.\end{gathered} \tag{2.11}$$

We seek the function ρ in a series expansion in ε:

$$\rho = \sum_{k=1}^{\infty} \rho_k \varepsilon^k. \tag{2.12}$$

Substituting (2.12) into (2.11), we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\partial^m u_n (r_2,\ \theta)}{\partial r^m} \left(\sum_{k=1}^{\infty} \rho_k \varepsilon^k\right)^m \varepsilon^n = K \sum_{k=1}^{\infty} \rho_k \varepsilon^k; \tag{2.13}$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\partial^{m+1} u_n (r_2,\ \theta)}{\partial r^{m+1}} \left(\sum_{k=1}^{\infty} \rho_k \varepsilon^k\right)^m \varepsilon^n = -K. \tag{2.14}$$

We shall assume that the angle θ_* at which the plastic zone encloses the outer contour L_2 is related to the small parameter ε by the relationship

$$\theta_* \sim \varepsilon,\ \sin\theta_* \sim \varepsilon,\ \cos\theta_* \sim 1. \tag{2.15}$$

The sum $(K + \partial u_0/\partial r)$ enters expressions (2.13) and (2.14), and, taking account of (2.2), (2.4), and (2.15), has the form

$$\left(K + \frac{\partial u_0}{\partial r}\right) = G\omega_0\left(T_1 \sin^2\frac{\theta}{2} + T_2 \sin^2\theta\right) = G\omega_0\left(\left(\frac{1}{4}T_1 + T_2\right)\theta^2 - \left(\frac{T_1}{48} + \frac{T_2}{3}\right)\theta^4 + \ldots\right), \tag{2.16}$$

where

$$T_1 = 4\delta K_1;\ T_2 = 8\delta^2 K_2 r_2.$$

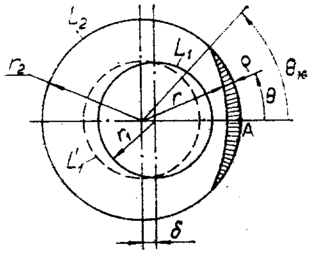

Fig. 1

Equating terms in identical powers of ε in (2.13) and (2.14) by taking account of (2.15) and (2.16), we obtain

$$u_0 = 0,\ u_1 = 0,\ u_2 = -\frac{\partial^2 u_0}{\partial r^2}\frac{\rho_1^2}{2} + \frac{\partial u_1}{\partial r}\rho_1,$$

$$u_3 = G\omega_0\rho_1\left(\frac{1}{4}T_1 + T_2\right)\frac{\theta^2}{\varepsilon^2} - \frac{\partial^2 u_0}{\partial r^2}\rho_1\rho_2 + \frac{\partial^3 u_0}{\partial r^3}\frac{\rho_1^3}{3!} + \frac{\partial u_1}{\partial r}\rho_2 - \frac{\partial^2 u_1}{\partial r^2}\frac{\rho_1^2}{2} + \frac{\partial u_2}{\partial r}\rho_1,\ u_4$$

$$= G\omega_0\rho_2\left(\frac{T_1}{4} + T_2\right)\frac{\theta^2}{\varepsilon^2} - \frac{\partial^2 u_0}{\partial r^2}\frac{(\rho_2^2 + 2\rho_1\rho_3)}{2} + \frac{\partial^3 u_0}{\partial r^3}\frac{\rho_1^2\rho_3}{2} - \frac{\partial^4 u_0}{\partial r^4}\frac{\rho_1^4}{4!} + \frac{\partial u_1}{\partial r}\rho_3 - \frac{\partial^2 u_1}{\partial r^2}\rho_1\rho_2 + \frac{\partial^3 u_1}{\partial r^3}\frac{\rho_1^3}{3!} + \frac{\partial u_2}{\partial r}(\rho_1 + \rho_2) + \frac{\partial^2 u_2}{\partial r^2}\frac{\rho_1^2}{2} - \frac{\partial u_3}{\partial r}\rho_1; \quad (2.17)$$

$$\rho_1 = \frac{\partial u_1}{\partial r}\left(\frac{\partial^2 u_0}{\partial r^2}\right)^{-1},\ \rho_2 = \left(G\omega_0\left(\frac{T_1}{4} + T_2\right)\frac{\theta^2}{\varepsilon^2} + \frac{\partial^3 u_0}{\partial r^3}\frac{\rho_1^2}{2} - \frac{\partial^2 u_1}{\partial r^2}\rho_1\right.$$
$$\left. + \frac{\partial u_2}{\partial r}\right)\left(\frac{\partial^2 u_0}{\partial r^2}\right)^{-1},\ \rho_3 = \left(\frac{\partial^3 u_0}{\partial r^3}\rho_1\rho_2 - \frac{\partial^4 u_0}{\partial r^4}\frac{\rho_1^3}{3!} - \frac{\partial^2 u_1}{\partial r^2}\rho_2 + \frac{\partial^3 u_1}{\partial r^3}\frac{\rho_1^2}{2} - \frac{\partial^2 u_2}{\partial r^2}\rho_1 + \frac{\partial u_3}{\partial r}\right)$$
$$\times\left(\frac{\partial^2 u_0}{\partial r^2}\right)^{-1},\ \rho_4 = \left(-G\omega_0\left(\frac{T_1}{48} + \frac{T_2}{3}\right)\frac{\theta^4}{\varepsilon^4} + \frac{\partial^3 u_0}{\partial r^3}\frac{\rho_2^2 + 2\rho_1\rho_3}{2} - \frac{\partial^4 u_0}{\partial r^4}\frac{\rho_1^2\rho_3}{2} + \frac{\partial^5 u_0}{\partial r^5}\frac{\rho_1^4}{4!}\right.$$
$$\left. - \frac{\partial^2 u_1}{\partial r^2}\rho_3 + \frac{\partial^3 u_1}{\partial r^3}\rho_1\rho_2 - \frac{\partial^4 u_1}{\partial r^4}\frac{\rho_1^3}{3!} - \frac{\partial^2 u_2}{\partial r^2}(\rho_1 + \rho_2) + \frac{\partial^3 u_2}{\partial r^3}\frac{\rho_1^2}{2} - \frac{\partial^2 u_3}{\partial r^2}\rho_1 + \frac{\partial u_4}{\partial r}\right)\left(\frac{\partial^2 u_0}{\partial r^2}\right)^{-1}. \quad (2.18)$$

The boundary conditions for u_n are obtained on the outer contour in the plastic zone from (2.17), and (2.18) permits determination of the plastic zone thickness ρ. Upon the development of a plastic zone on the outer contour L_2, the expression of the Bredt theorem for the inner contour L_1 with orifice area F_1 has the form

$$\int_{L_1}\left(-\frac{\partial u}{\partial r}r\,d\theta + \frac{1}{r}\frac{\partial u}{\partial\theta}dr\right) = 2G\omega F_1. \quad (2.19)$$

Taking account of (2.6) and equating terms in (2.19) for identical powers of ε, we obtain

$$P(u_0) = 2G\omega_0 F_1,\ P(u_1) = 0,\ P(u_2) = 2G\omega_0 F_1,\ P(u_n) = 0,\ n = 3, 4\ldots, \quad (2.20)$$

where

$$P(u_n) = \int_{L_1}\left(-\frac{\partial u_n}{\partial r}r\,d\theta + \frac{1}{r}\frac{\partial u_n}{\partial\theta}dr\right).$$

We seek the solution u_n in the form of the series

$$u_n = \sum_{l=0}^{\infty} u_{nl}\delta^l,\quad n = 1, 2\ldots \quad (2.21)$$

Let us expand u_n around the contour L_1' which is concentric to L_2:

$$u_n = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{\partial^m u_{nl}}{\partial r^m}\frac{\left(\delta\cos\theta - \frac{\delta^2}{2r_1}\sin^2\theta\right)^m\delta^l}{m!},\quad n = 1, 2\ldots \quad (2.22)$$

Equating terms in identical powers of δ in (2.20), and taking account of (2.21) and (2.22), we obtain

$$\int_0^{2\pi} \frac{\partial u_{n0}}{\partial r} r_1 d\theta = 0,$$
$$\int_0^{2\pi} \left(\left(\frac{\partial^2 u_{n0}}{\partial r^2} r_1 + \frac{\partial u_{n0}}{\partial r} \right) \cos\theta + \frac{\partial u_{n1}}{\partial r} r_1 + \frac{\partial u_{n0}}{\partial \theta} \frac{\sin\theta}{r_1} \right) d\theta = 0, \; n = 1, 3, 4 \ldots \tag{2.23}$$

Equations (2.23), except the first, are used for u_2, since according to (2.20) it will have the form

$$\int_0^{2\pi} -\frac{\partial u_{20}}{\partial r} r_1 d\theta = 2G\omega_0 F_1. \tag{2.24}$$

On the inner contour L_1 the function u has the constant value

$$u = c. \tag{2.25}$$

We represent the constant c in the form

$$c = \sum_{n=0}^{\infty} c_n \varepsilon^n. \tag{2.26}$$

Taking account of (2.26), we equate terms in identical powers of ε in (2.25) and obtain

$$u_n = c_n, \; n = 0, 1, 2 \ldots \tag{2.27}$$

We represent c_n in the form

$$c_n = \sum_{l=0}^{\infty} c_{nl} \delta^l. \tag{2.28}$$

We expand the value of u_n on L_1 around L_1' which is concentric with L_2, and equating terms of identical powers of δ, by taking account of (2.22) and (2.28), we obtain the relationships

$$u_{n0} = c_n, \; u_{n1} = c_{n1} - \frac{\partial u_{n0}}{\partial r} \cos\theta, \; n = 1, 2 \ldots \tag{2.29}$$

The boundary conditions for u on the outer contour in the elastic domain are

$$u = 0. \tag{2.30}$$

The boundary conditions for u_n are determined from (2.30)

$$u_n = 0, \; u_{nl} = 0, \; l, \; n = 0, 1, 2 \ldots \tag{2.31}$$

Taking account of (2.21), Eq. (2.9) for u_n dissociates into a system of equations for u_{nl}:

$$\Delta u_{nl} = 0, \; n = 1, 3, 4 \ldots, \; l = 0, 1, 2 \ldots,$$
$$\Delta u_{20} = -2G\omega_0, \; \Delta u_{2l} = 0, \; l = 1, 2, 3 \ldots \tag{2.32}$$

The differential equations and boundary conditions obtained permit separation of the problem of determining the function u into a number of sequential problems on finding the functions u_n, u_{nl} by solving (2.9), (2.32) with the boundary conditions (2.17), (2.31), (2.20), (2.27) for u_n and (2.23), (2.24), (2.29) for u_{nl}.

3. Let us successively consider the solution of the problem of finding u_n, u_{nl}. The function u_1 has the following boundary-value problem for the determination

$$\Delta u_1 = 0, \; u_1 = 0 \text{ on } L_2,$$
$$u_1 = c_1, \; \int_{L_1} -\frac{\partial u_1}{\partial r} r d\theta + \frac{1}{r} \frac{\partial u_1}{\partial \theta} dr = 0 \text{ on } L_1. \tag{3.1}$$

Taking account of (2.18), we obtain from (3.1)

$$u_1 = 0, \; \rho_1 = 0. \tag{3.2}$$

The determination of u_1, ρ_1 permits finding the boundary conditions for u_2 on the contour L_2 for $-\theta_* \le \theta \le \theta_*$. It follows from (2.17) that $u_2 = 0$ on the contour L_2 for $-\theta_* \le \theta \le \theta_*$. Then the function u_2 has the following boundary-value problem for its determination:

$$\Delta u_2 = -2\omega_0 G, \; u_2 = 0 \text{ on } L_2,$$
$$u_2 = c_2, \; \int_{L_1} -\frac{\partial u_2}{\partial r} r d\theta + \frac{1}{r}\frac{\partial u_2}{\partial \theta} dr = 2\omega_0 GF_1 \text{ on } L_1. \tag{3.3}$$

The boundary-value problem (3.3) for u_2 agrees with the problem of elastic torsion of an eccentric ring whose solution is (2.2). Therefore,

$$u_2 = u_0. \tag{3.4}$$

Taking account of (3.2) and (3.4), we determine ρ_2 from (2.18)

$$\rho_2 = \left(G\omega_0\left(\frac{T_1}{4} + T_2\right)\frac{\theta^2}{\varepsilon^2} + \frac{\partial u_0}{\partial r}\right)\left(\frac{\partial^2 u_0}{\partial r^2}\right)^{-1},$$

from which we have after substituting $\partial u_0/\partial r$, $\partial^2 u_0/\partial r^2$ and taking account of (2.15)

$$\rho_2 = G\omega_0\left(\frac{T_1}{4} + T_2\right)(\theta_*^2 - \theta^2)\,\varepsilon^{-2} L_2^{-1}; \tag{3.5}$$

$$\theta_* = \varepsilon \sqrt{\frac{L_1}{G\omega_0\left(\frac{T_1}{4} + T_2\right)}}, \tag{3.6}$$

where

$$L_1 = G\omega_0(r_2 + 2\delta K_1 + 4\delta^2 K_2 r_2);$$
$$L_2 = G\omega_0\left(1 - \frac{2\delta K_1}{r_2} - 4\delta^2 K\right).$$

The angle θ_* at which the contour L_2 is enclosed by a plastic zone is determined from the condition $\rho_2 = 0$ for $\theta = \theta_*$. According to (3.2), the boundary-value problem of determining u_3 agrees with the conditions for finding u_1, from which we obtain

$$u_3 = 0, \; \rho_3 = 0. \tag{3.7}$$

Let us turn to a determination of the functions $u_4(r, \theta)$ and ρ_4. Taking account of (3.2), (3.4), (3.5), and (3.7), we obtain a value of the function u_4 on the contour L_2 in the plastic domain from (2.17)

$$u_4 = G\omega_0\left(\frac{T_1}{4} + T_2\right)\frac{\theta^2}{\varepsilon^2} - \frac{\partial^2 u_0}{\partial r^2}\frac{\rho_2^2}{2} + \frac{\partial u_0}{\partial r}\rho_2,$$

from which we have after manipulation

$$u_4 = -L_3(\theta_*^2 - \theta^2)^2, \tag{3.8}$$

where $L_3 = \dfrac{\left(G\omega_0\left(\frac{T_1}{4} + T_2\right)\right)^2}{2\varepsilon^4 L_2}$. The function u_4 is determined on the contour L_2 in the plastic domain for the angle of enclosure (3.6) and is found from the solution of the following boundary-value problem

$$\Delta u_4 = 0, \; u_4 = \begin{cases} -L_3(\theta_*^2 - \theta^2)^2 & \text{on } L_2, \; -\theta_* \leqslant \theta \leqslant \theta_*, \\ 0 & \text{on } L_2, \; \theta_* < \theta < 2\pi - \theta_*, \\ c_4 & \text{on } L_1, \end{cases} \tag{3.9}$$
$$\int_{L_1} -\frac{\partial u_4}{\partial r} r d\theta + \frac{1}{r}\frac{\partial u_4}{\partial \theta} dr = 0.$$

In conformity with (2.21)-(2.23), (2.28), (2.29), (2.31), the problem of finding u_4 in the domain bounded by the contours L_1' and L_2 (an eccentric ring) can be reduced to the successive solution of boundary-value problems in the domain bounded by the contours L_1' and L_2 (concentric ring), by the representation

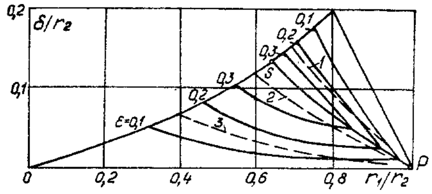

Fig. 2

$$u_4 = u_{40} + \delta u_{41} + \ldots \tag{3.10}$$

Let us limit ourselves to determining u_{40}, u_{41}. For a function given continuously on the whole contour L_2 we expand the functions u_{40}, u_{41} given on $-\theta_* \leq \theta \leq \theta_*$ in a Fourier series. By using (3.8) we designate the boundary conditions for u_{40}, u_{41} on L_2 for $-\theta_* \leq \theta \leq \theta_*$, in such a way that the functions u_{40}, u_{41} and their derivatives with respect to r and θ obtained as a result of solving the boundary-value problem would have convergent series in the whole elastic domain and on its boundaries

$$u_{40} = -L_3(\theta_*^2 - \theta^2)^2, \; u_{41} = 0. \tag{3.11}$$

We reduce finding u_{40} to the solution of the following boundary-value problem by following (3.9)-(3.11)

$$\Delta u_{40} = 0, \; u_{40} = \begin{cases} E + \sum_{k=1}^{\infty} E_k \cos k\theta \text{ on } L_2, \\ c_{40} \text{ on } L_1', \end{cases} \tag{3.12}$$

$$\int_0^{2\pi} \frac{\partial u_{40}}{\partial r} r_1 d\theta = 0,$$

where

$$E = -\frac{8}{15}\frac{\theta_*^5}{\pi} L_3; \; E_k = \frac{16}{\pi} L_3 \left(\frac{3\theta_*}{k^4}\cos k\theta_* + \left(\frac{\theta_*^2}{k^3} - \frac{3}{k^5}\right)\sin k\theta_*\right).$$

We seek the solution of (3.12) in the form

$$u_{40} = R + R_0 \ln r + \sum_{k=1}^{\infty} (R_k r^k + R_k' r^{-k}) \cos k\theta. \tag{3.13}$$

We obtain the value of the function u_{40} from (3.12) and (3.13)

$$u_{40} = E + \sum_{k=1}^{\infty} R_k r^k \left(1 - \left(\frac{r_1}{r}\right)^{2k}\right) \cos k\theta, \tag{3.14}$$

where

$$R_k = \frac{E_k r_2^k}{r_2^{2k} - r_1^{2k}}.$$

To determine u_{41} it is necessary to solve the boundary-value problem

$$\Delta u_{41} = 0, \; u_{41} = \begin{cases} 0 \quad \text{on } L_2, \\ c_{41} - \frac{\partial u_{40}}{\partial r}\cos\theta \text{ on } L_1', \end{cases} \tag{3.15}$$

$$\int_0^{2\pi} \left(\left(\frac{\partial^2 u_{40}}{\partial r^2} r_1 + \frac{\partial u_{40}}{\partial r}\right)\cos\theta + \frac{\partial u_{41}}{\partial r} r_1 + \frac{\partial u_{40}}{\partial \theta}\frac{\sin\theta}{r_1}\right) d\theta = 0.$$

We seek the solution in the form (3.13). From (3.13) and (3.15) we find

$$u_{41} = \sum_{k=1}^{\infty} W_k r^k \left(1 - \left(\frac{r_2}{r}\right)^{2k}\right) \cos k\theta, \tag{3.16}$$

where

$$W_k = \frac{R_{k-1}(k-1)r_1^{2k-2} + R_{k+1}(k+1)r_1^{2k}}{r_2^{2k} - r_1^{2k}}.$$

By obtaining the value of the function u_4 and its derivative $\partial u_4/\partial r$ from (3.10), (3.14), and (3.16), we determine ρ_4 from (2.18) in the form

$$\rho_4 = \frac{1}{2}\left(-G\omega_0\left(\frac{T_1}{48} + \frac{T_2}{3}\right)\frac{\theta^4}{\varepsilon^4} + \frac{\partial^3 u_0}{\partial r^2}\frac{\rho_2^2}{2} - \frac{\partial^2 u_0}{\partial r^2}\rho_2 + \frac{\partial u_4}{\partial r}\right)\left(\frac{\partial^3 u_0}{\partial r^3}\right)^{-1}. \quad (3.17)$$

According to (2.12), by using (3.5) and (3.17) we obtain the thickness of the plastic zone along the normal to the contour L_2 after the second approximation $\rho = \varepsilon^2\rho_2 + \varepsilon^4\rho_4$ and the location of the elastic–plastic boundary thereby (see Fig. 1). We obtain the stress function in the elastic domain from (2.7), (3.2), (3.4), (3.7), (3.14) and (3.16)

$$u = (1 + \varepsilon^2)u_0 + \varepsilon^4(u_{40} + \delta u_{41}),$$

which, according to (1.1), permits determination of the stress state in the elastic domain.

The solution constructed satisfies the exact equations of the theory of ideal plasticity in the plastic domain, and the exact equations of the theory of elasticity in the elastic domain. For a limited number of approximations, the perturbations method reduces consequently to approximate satisfaction of the boundary conditions and the connection conditions on the elastic–plastic boundary.

The conditions on the outer contour L_2 are satisfied exactly, and approximately on the inner contour L_1 and the elastic–plastic boundary.

Hence, the accuracy of the solution obtained can be assessed from the magnitudes of the relative residuals $(\tau_\theta - K)/K$, τ_r/K on the elastic–plastic boundary as well as from the magnitude of the relative residual of the boundary conditions on the inner contour τ_n/K, where τ_n is the stress normal to the inner contour L_1.

The domain of the parameters δ/r_2, r_1/r_2 that govern the geometry of the pipe cross section is represented in Fig. 2. For each point of the domain $(\delta/r_1, r_1/r_2)$ the greatest parameters ε are indicated when the residuals do not exceed 1%.

The curve PS, which is the locus of points of intersection of the curves ε = const, is presented in Fig. 2. The curves ε = const when the plastic flow reaches the contour L_1 is located above PS. For large ε the method of solution being applied cannot be used. Below PS are the curves ε = const when the plastic flow does not reach the inner contour L_1.

The parameter ε varies along PS from $\varepsilon = 0$ at the point P to $\varepsilon = 0.38$ at the point S, where ε is greatest for the whole domain extracted. Dashed curves 1-3 are presented in Fig. 2, along which the angle at which a plastic zone enclosed the outer contour L_2 is constant at 15, 50, 45°, respectively.

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